IRREDUCIBILITY OF THE WEYL REPRESENTATION

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Abstract. Let H be a separable Hilbert space and denote its symmetric Fock space by $\Gamma_s(H)$. Let $\mathcal{A}$ be the linear span of $\{W(\xi) : \xi \in H\}$, where $\{W(\xi) : \xi \in H\}$ stands for the collection of Weyl operators on $\Gamma_s(H)$. The purpose of this note is to record a proof of the fact that $\mathcal{A}$ is irreducible, i.e. the commutant of $\mathcal{A}$ is trivial. Consequently, $\mathcal{A}$ is weakly (and strongly) dense in $B(\Gamma_s(H))$. This fact is crucial to construct CCR flows in the theory of E_0-semigroups. Proofs of this fact can be found in [2] and [1]. This note grew out of the author's attempt to understand the proofs presented in those two books. I believe that graduate students working in E_0-semigroups, quantum probability and related topics will find this note to be of use.

1. Symmetric Fock space, exponential vectors and the Weyl operators

Let H be a separable Hilbert space. Set $H^{\otimes 0} := \mathbb{C}$ and for $n \geq 1$, set

$$H^{\otimes n} := \underbrace{H \otimes H \otimes \cdots \otimes H}_{\text{n times}}.$$

For $n \geq 1$, let S_n be the permutation group on n symbols. For $\sigma \in S_n$, let U_σ be the operator on $H^{\otimes n}$ defined by

$$U_\sigma(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \xi_{\sigma(1)} \otimes \xi_{\sigma(2)} \otimes \cdots \otimes \xi_{\sigma(n)}.$$

Clearly, for $\sigma \in S_n$, U_σ is a unitary operator.

Define

$$H_s^{\otimes n} := \{\xi \in H^{\otimes n} : U_\sigma(\xi) = \xi \text{ for every } \sigma \in S_n\}.$$

Set $H_s^{\otimes 0} := \mathbb{C}$. The direct $\bigoplus_{n \geq 0} H_s^{\otimes n}$ is called the symmetric Fock space and is denoted $\Gamma_s(H)$. For $n \geq 1$, let $P_n : H^{\otimes n} \to H^{\otimes n}$ be the orthogonal projection onto $H_s^{\otimes n}$. Then,

$$P_n := \frac{1}{n!} \sum_{\sigma \in S_n} U_\sigma.$$

For $\xi \in H$, let $\xi^{\otimes n} := \underbrace{\xi \otimes \xi \otimes \cdots \otimes \xi}_{\text{n times}}$ if $n \geq 1$ and set $\xi^{\otimes n} = 1$ if $n = 0$.

Lemma 1.1. *For every $n \geq 0$, the set $\{\xi^{\otimes n} : \xi \in H\}$ is a total subset of $H_s^{\otimes n}$.*

Proof. We can assume that $n \geq 2$. Let D be the closure of the linear span of $\{\xi^{\otimes n} : \xi \in H\}$. Note that $\{P_n(\xi_1 \otimes \cdots \otimes \xi_n) : \xi_1, \xi_2, \cdots, \xi_n \in H\}$ is a total set in $H_s^{\otimes n}$. Thus, it suffices to show that $P_n(\xi_1 \otimes \cdots \otimes \xi_n) \in D$ for every $\xi_1, \xi_2, \cdots, \xi_n \in H$. To that effect, let $\xi_1, \xi_2, \cdots, \xi_n \in H$ be given. Observe that up to a scalar factor, $P_n(\xi_1 \otimes \xi_2 \cdots \otimes \xi_n)$ coincides with the coefficient of $t_1 t_2 \cdots t_n$ of the 'polynomial' $(t_1\xi_1 + t_2\xi_2 + \cdots + t_n\xi_n)^{\otimes n}$ which takes values in D. The proof follows. □.

For $\xi \in H$, let

$$e(\xi) := \bigoplus_{n \geq 0} \frac{\xi^{\otimes n}}{\sqrt{n!}}.$$

The collection $\{e(\xi) : \xi \in H\}$ is called the set of exponential vectors. Note that for $\xi, \eta \in H$,

$$\langle e(\xi) | e(\eta) \rangle = e^{\langle \xi | \eta \rangle}.$$

Lemma 1.2. *The set $\{e(\xi) : \xi \in H\}$ is total in $\Gamma_s(H)$.*

Proof. Let D be the closure of the linear span of $\{e(\xi) : \xi \in H\}$. Thanks to Lemma 1.1, it suffices to prove that $\xi^{\otimes n} \in D$ for every $\xi \in H$. Let $\xi \in H$ be given. Define a map $f : \mathbb{R} \to D$ by $f(t) := e(t\xi)$. The map f is analytic. Note that up to a scalar factor,

$$\xi^{\otimes n} := f^{(n)}(0).$$

The proof follows. □

Remark 1.3. *The vector $e(0) = 1 \oplus 0 \oplus 0 \cdots$ is usually called the vacuum vector.*

Remark 1.4. *We will repeatedly make use of the following. Suppose H_1 and H_2 are Hilbert spaces and S_1 and S_2 are total subsets of H_1 and H_2 respectively. Let $\phi : S_1 \to S_2$ be a map such that $\langle \phi(x) | \phi(y) \rangle = \langle x | y \rangle$ for $x, y \in S_1$. Then, there exists a unique isometry $V : H_1 \to H_2$ which extends ϕ. Moreover if ϕ is a bijection, then the isometry V is a unitary.*

The following statements follow directly from the previous remark and the totality of the exponential vectors.

(1) Let $\xi \in H$ be given. Then, there exists a unique unitary operator on $\Gamma_s(H)$, denoted $W(\xi)$, such that

$$W(\xi)e(\eta) = e^{-\frac{||\xi||^2}{2} - \langle \eta | \xi \rangle} e(\eta + \xi).$$

The collection $\{W(\xi) : \xi \in H\}$ is called the set of *Weyl operators*. Moreover, the Weyl operators satisfy the following relations called *the canonical commutation relations* abbreviated as CCR. For $\xi, \eta \in H$,

$$W(\xi)W(\eta) = e^{iIm\langle \xi | \eta \rangle} W(\xi + \eta)$$

where $Im(\langle \xi|\eta\rangle)$ denotes the imaginary part of $\langle \xi|\eta\rangle$. In particular, for $\xi \in H$, $W(\xi)^* = W(-\xi)$.

(2) Let H_1 and H_2 be Hilbert spaces. Then, there exists a unique unitary operator from $\Gamma_s(H_1 \oplus H_2) \to \Gamma_s(H_1) \otimes \Gamma_s(H_2)$ which maps $e(\xi_1 \oplus \xi_2)$ to $e(\xi_1) \otimes e(\xi_2)$. We always identify $\Gamma_s(H_1 \oplus H_2)$ with $\Gamma_s(H_1) \otimes \Gamma_s(H_2)$ via this identification. Under this identification, note that for $\xi_1 \in H_1$ and $\xi_2 \in H_2$,
$$W(\xi_1 \oplus \xi_2) = W(\xi_1) \otimes W(\xi_2).$$

Let $\mathcal{A}$ be the linear span of $\{W(\xi) : \xi \in H\}$. From the CCR relations, it is clear that $\mathcal{A}$ is a unital *-subalgebra of $B(\Gamma_s(H))$.

The main aim of this note is to prove the following theorem.

Theorem 1.5. *The *-subalgebra $\mathcal{A}$ acts irreducibly on $\Gamma_s(H)$, i.e. $\mathcal{A}' = \mathbb{C}$. Consequently, $\mathcal{A}$ is weakly (and strongly) dense in $B(\Gamma_s(H))$.*

2. Proof of the main theorem

First, we prove Thm. 1.5 when H is finite dimensional. Thus, assume $H = \mathbb{C}^n$. Since
$$\Gamma_s(\mathbb{C}^n) = \bigotimes_{i=1}^{n} \Gamma_s(\mathbb{C})$$
and
$$W(z_1, z_2, \cdots, z_n) = \bigotimes_{i=1}^{n} W(z_i),$$
it suffices to prove Thm. 1.5 when H is one dimensional.

Let us concentrate on the 1-dimensional case. One of the key steps involved is to realise the Weyl commutation relation from the CCR relation. It is also possible to realise the CCR relation from the Weyl commutation relation which we do not need.

For $s, t \in \mathbb{R}$, set $U_s := W(s)$ and $V_t := W(\frac{it}{2})$. Then, using the CCR relation twice, we get
$$U_sV_t = e^{its}V_tU_s$$
for $s, t \in \mathbb{R}$. The above relation is called *the Weyl commutation relation*. It is better at this point to make a formal definition.

Definition 2.1. *Let K be a Hilbert space and let $U := \{U_s\}_{s\in\mathbb{R}}$ and $V := \{V_t\}_{t\in\mathbb{R}}$ be two strongly continuous 1-parameter group of unitaries on K. We say that the pair (U, V) is a representation of the Weyl commutation relation if*
$$U_sV_t = e^{its}V_tU_s$$
for every $s, t \in \mathbb{R}$.

The representation (U,V) is said to be irreducible if and only if $\{U_s, V_t : s,t \in \mathbb{R}\}' = \mathbb{C}$.

Suppose (U,V) and $(\widetilde{U},\widetilde{V})$ are two representations of the Weyl commutation relation. Then, we say that (U,V) and $(\widetilde{U},\widetilde{V})$ are unitarily equivalent if there exists a unitary $X : K \to \widetilde{K}$ such that

$$XU_sX^* = \widetilde{U}_s;\;\; XV_tX^* = \widetilde{V}_t.$$

Here, K is the Hilbert space on which (U,V) acts and $\widetilde{K}$ is the Hilbert space on which $(\widetilde{U},\widetilde{V})$ acts.

The Stone-von Neumann theorem is stated below. Strictly speaking, we do not need the following theorem for the proof of Thm. 1.5. However, the reader should be aware of it.

Theorem 2.2 (Stone-von Neumann)**.** *Up to unitary equivalence, there is one and only irreducible representation of the Weyl commutation relation.*

Let us give another 'natural choice' for the Weyl commutation relation.

Example 2.3. *Let $K := L^2(\mathbb{R})$. For $s \in \mathbb{R}$, let U_s be the unitary operator on K defined by*

$$U_s(\xi)(x) = f(x-s).$$

For $t \in \mathbb{R}$, let V_t be the unitary operator on K defined by

$$V_t(\xi)(x) = e^{-itx}\xi(x).$$

We leave it to the reader to verify that (U,V) is a representation of the Weyl commutation relation. We call this pair (U,V) ***the canonical representation*** *of the Weyl commutation relation.*

Example 2.4. *Let $L := \Gamma_s(\mathbb{C})$. For $s \in \mathbb{R}$, let $\widetilde{U}_s := W(s)$ and for $t \in \mathbb{R}$, let $\widetilde{V}_t := W(\frac{it}{2})$. Then, $(\widetilde{U},\widetilde{V})$ is a representation of the Weyl commutation relation and we call the pair $(\widetilde{U},\widetilde{V})$* ***the Weyl representation*** *of the Weyl commutation relation.*

What we ought to prove is that the Weyl representation is irreducible. We prove this in steps.

1 First, we prove that the canonical representation is irreducible.

2 Next, we compare the Weyl representation with the canonical representation and show that they are unitarily equivalent. For, the Stone-von Neumann theorem says that if Thm. 1.5 is true, then the Weyl representation and the canonical representation are indeed equivalent. So, we are, in some sense, compelled to compare the two representations. Once we show that the Weyl representation and the canonical representation are unitarily equivalent, the irreducibility of the Weyl representation follows from that of the canonical one.

Remark 2.5. *Let $(X, \mathcal{B})$ be a measurable space and let μ be a σ-finite measure on X. For $\phi \in L^\infty(X,\mu)$, let M_ϕ be the bounded linear operator on $L^2(X,\mu)$ defined by*

$$M_\phi(\xi)(x) = \phi(x)\xi(x)$$

for $\xi \in L^2(X,\mu)$.

The map

$$L^\infty(X,\mu) \ni \phi \to M_\phi \in B(L^2(X,\mu))$$

is continuous when $L^\infty(X,\mu)$ is given the weak *-topology (after identifying $L^\infty(X,\mu)$ with the dual of $L^1(X,\mu)$) and $B(L^2(X,\mu))$ is given the weak topology. Moreover,

$$\{M_\phi : \phi \in L^\infty(X,\mu)\}' = \{M_\phi : \phi \in L^\infty(X,\mu)\}.$$

Proposition 2.6. *The canonical representation of the Weyl commutation relation is irreducible*

Proof. Let (U, V) be the canonical representation. Let $T \in \{U_s, V_t : s,t \in \mathbb{R}\}'$ be given.

For $t \in \mathbb{R}$, let $\phi_t \in L^\infty(\mathbb{R})$ be defined by $\phi_t(x) = e^{-itx}$. Note that $V_t = M_{\phi_t}$. We claim that the linear span of $\{\phi_t : t \in \mathbb{R}\}$ is weak *-dense in $L^\infty(\mathbb{R})$. Suppose not. Then, there exists a non-zero $f \in L^1(\mathbb{R})$ such that for every $t \in \mathbb{R}$,

$$\int e^{-itx} f(x)dx = 0.$$

In other words, the Fourier transform of f is zero which forces $f = 0$. This is a contradiction to the fact that $f \neq 0$. Hence, $\{\phi_t : t \in \mathbb{R}\}$ is weak *-dense in $L^\infty(\mathbb{R})$.

Since T commutes with M_{ϕ_t} for every $t \in \mathbb{R}$ and $\{\phi_t : t \in \mathbb{R}\}$ is weak *-dense in $L^\infty(\mathbb{R})$, it follows from Remark 2.5 that T commutes with M_ϕ for every $\phi \in L^\infty(\mathbb{R})$. Again, thanks to Remark 2.5, it follows that there exists $\phi \in L^\infty(\mathbb{R})$ such that $T = M_\phi$. Since T is positive, we can choose ϕ to be non-negative.

The fact that $U_s T U_s^* = T$ for every $s \in \mathbb{R}$ translates to the fact that for every $s \in \mathbb{R}$, $\phi(x+s) = \phi(x)$ for almost all $x \in \mathbb{R}$. Let $\omega_\phi : C_c(\mathbb{R}) \to \mathbb{C}$ be defined by

$$\omega_\phi(f) = \int f(t)\phi(t)dt.$$

To show that ϕ is a scalar, it suffices to show that ω_ϕ is a scalar multiple of the linear functional $I : C_c(\mathbb{R}) \to \mathbb{C}$ defined by the equation

$$I(f) = \int f(t)dt.$$

Let $g \in C_c(\mathbb{R})$ be such that $I(g) = 0$. Choose $f \in C_c(\mathbb{R}^n)$ such that $\int f(t)dt = 1$. Calculate as follows to observe that

$$\begin{aligned}\omega_\phi(g) &= \int f(s)(\int g(t)\phi(t)dt)ds \\ &= \int f(s)(\int g(t)\phi(t-s)dt)ds \\ &= \int g(t)(\int f(s)\phi(t-s)ds)dt \\ &= \int g(t)(\int f(s)\phi(-s)ds)dt \\ &= (\int g(t)dt)(\int f(s)\phi(-s)ds) \\ &= 0.\end{aligned}$$

Hence, $Ker(I) \subset Ker(\omega_\phi)$. This shows that ω_ϕ is a scalar multiple of I. Hence, ϕ is a scalar and consequently T is a scalar. The proof is now complete. □

Next, we compare the Weyl representation with the canonical one. Let $(\widetilde{U}, \widetilde{V})$ be the Weyl representation and let (U, V) be the canonical representation. Observe the following.

(1) The vacuum vector $\Omega := e(0)$ is cyclic for the Weyl representation by which we mean that $\{\widetilde{U}_s\widetilde{V}_t\Omega : s, t \in \mathbb{R}\}$ is total in $\Gamma_s(\mathbb{C})$. Consequently, the Weyl representation is completely determined by 'the state' given by the vacuum vector.
(2) As far as the canonical representation is concerned, being irreducible, every unit vector is cyclic. So, if we could find a unit vector $\xi \in L^2(\mathbb{R})$ such that 'the state' determined by ξ in the canonical representation coincides with 'the state' determined by the vacuum vector in the Weyl representation, then we are done.

The reader should compare the above statements with the usual statements that one make after learning about the GNS construction. The reader might wonder where the states are defined. Does there exist a C^*-algebra on which the two states mentioned above live? The answer is yes; although, we will not discuss this issue. The interested reader is recommended to read the operator algebraic proof of the Stone-von Neumann theorem that can be found, for instance, in [3] and figure out the C^*-algebra on her own.

Coming back, what we seek is a unit vector $\xi \in L^2(\mathbb{R})$ such that for every $s, t \in \mathbb{R}$,

$$\langle \widetilde{U}_s\widetilde{V}_t\Omega|\Omega\rangle = \langle U_sV_t\xi|\xi\rangle.$$

Assume for the moment that this is achieved. Calculate as follows to observe that for $s_1, s_2, t_1, t_2 \in \mathbb{R}$,

$$\langle U_{s_1}V_{t_1}\xi|U_{s_2}V_{t_2}\xi\rangle = \langle V_{t_2}^*U_{s_2}^*U_{s_1}V_{t_1}\xi|\xi\rangle$$

$$
\begin{aligned}
&= \langle V_{-t_2} U_{s_1 - s_2} V_{t_1} \xi | \xi \rangle \\
&= e^{it_2(s_1 - s_2)} \langle U_{s_1 - s_2} V_{t_1 - t_2} \xi | \xi \rangle \\
&= e^{it_2(s_1 - s_2)} \langle \widetilde{U}_{s_1 - s_2} \widetilde{V}_{t_1 - t_2} \Omega | \Omega \rangle \\
&= e^{it_2(s_1 - s_2)} \langle \widetilde{U}_{s_1 - s_2} \widetilde{V}_{-t_2} \widetilde{V}_{t_1} \Omega | \Omega \rangle \\
&= \langle \widetilde{V}_{-t_2} \widetilde{U_{-s_2}} \widetilde{U}_{s_1} \widetilde{V}_{t_1} \Omega | \Omega \rangle \\
&= \langle \widetilde{U}_{s_1} \widetilde{V}_{t_1} \Omega | \widetilde{U}_{s_2} \widetilde{V}_{t_2} \Omega \rangle .
\end{aligned}
$$

Thanks to the above calculation, the totality of $\{U_s V_t \xi : s, t \in \mathbb{R}\}$, the totality of $\{\widetilde{U}_s \widetilde{V}_t \Omega : s, t \in \mathbb{R}\}$ and Remark 1.4, it follows that there exists a unique unitary operator $X : L^2(\mathbb{R}) \to \Gamma_s(\mathbb{C})$ such that

$$X(U_s V_t \xi) = \widetilde{U}_s \widetilde{V}_t \Omega.$$

By calculating the action of the relevant operators on the total set and by making use of the Weyl commutation relation, we see that $XU_sX^* = \widetilde{U}_s$ and $XV_tX^* = \widetilde{V}_t$ for every $s, t \in \mathbb{R}$. We leave this verification to the reader. This proves that the Weyl representation and the canonical representation are unitarily equivalent.

How to find the desired vector $\xi \in L^2(\mathbb{R})$? The equation

$$\langle U_s V_t \xi | \xi \rangle = \langle \widetilde{U}_s \widetilde{V}_t \Omega | \Omega \rangle$$

translates to the equation

$$\int e^{-itx} \xi(x - s) \overline{\xi(x)} d\lambda(x) = e^{-\frac{t^2}{8}} e^{-\frac{s^2}{2}} e^{\frac{-its}{2}}. \tag{2.1}$$

The above equation needs to be satisfied for every $s, t \in \mathbb{R}$. If we substitute $s = 0$, we see that the Fourier transform of $|\xi|^2$ is a Gaussian and hence $|\xi|^2$ must be a Gaussian. Therefore, $|\xi|$ must be a Gaussian. Why not just take ξ to be a Gaussian and see what happens.

Let us recall the formula regarding the Gaussian that we must be able to work out on our own. I took it from Wikipedia. (The reader must do the necessary computation of the integral.) For $a > 0$, we have

$$\sqrt{\frac{a}{\pi}} \int e^{ibx} e^{-ax^2} dx = e^{-\frac{b^2}{4a}}. \tag{2.2}$$

It is appropriate at this point to scale the Lebesgue measure by a factor of $\sqrt{\frac{2}{\pi}}$. So, let $d\lambda(x) := \sqrt{\frac{2}{\pi}} dx$. The Hilbert space $L^2(\mathbb{R})$ is now $L^2(\mathbb{R}, d\lambda)$. Define $\xi \in L^2(\mathbb{R}, d\lambda)$ by

$$\xi(x) = e^{-x^2}.$$

Then, ξ is of unit norm. Using Eq. 2.2 and by elementary calculations, we see that Eq. 2.1 is satisfied for every $s, t \in \mathbb{R}$. The proof of Thm. 1.5 when H is finite dimensional is complete.

Next, we proceed towards proving Thm. 1.5 when H is infinite dimensional.

Notation: For a vector $u \in \Gamma_s(H)$, we denote the n-the component of u by u_n, i.e.

$$u = \bigoplus_{n=0}^{\infty} u_n.$$

Let $\{\xi_1, \xi_2, \cdots, \}$ be an orthonormal basis for H.

For $n \geq 1$, let $H_n := span\{\xi_1, \xi_2, \cdots, \xi_n\}$. Let $U_n : \Gamma_s(H_n) \otimes \Gamma_s(H_n^{\perp}) \to \Gamma_s(H)$ be the unique unitary such that

$$U_n(e(\xi) \otimes e(\eta)) = e(\xi + \eta).$$

Note that for $\xi \in H_n$ and $\eta \in H_n^{\perp}$, $U_n^* W(\xi + \eta) U_n = W(\xi) \otimes W(\eta)$.

Lemma 2.7. *Keep the foregoing notation. Let $m \geq 1$ be given. For $u \in \Gamma_s(H_m^{\perp})$, let x_k be the k-component of $U_m(e(0) \otimes u)$. Then, x_k is orthogonal to $\{\xi^{\otimes k} : \xi \in H_m\}$.*

Proof. It suffices to prove the assertion when $u = e(\eta)$ for some $\eta \in H_m^{\perp}$. Then, $x_k = \frac{\eta^{\otimes k}}{\sqrt{k!}}$ which is clearly orthogonal to $\{\xi^{\otimes k} : \xi \in H_m\}$. The proof is complete. $\square$

We are all set to prove Thm. 1.5.

Proof of Thm. 1.5. Let $T \in \{W(\xi) : \xi \in H\}'$. We claim that $x := Te(0)$ is a scalar multiple of $e(0)$. For $n \geq 1$, let x_n be the n-component of x. We need to prove that $x_n = 0$ for $n \geq 1$.

Let $n \geq 1$ be given. Since $\{\xi^{\otimes n} : \xi \in H_m, m \geq 1\}$ is total in $H_s^{\otimes n}$, it suffices to show that for every m and for every $\xi \in H_m$, $\langle x_n | \xi^{\otimes n} \rangle = 0$. Suppose $m \geq 1$ and $\xi \in H_m$. Note that $U_m^* T U_m \in \{W(u) \otimes 1 : u \in H_m\}'$. Thanks to the finite dimensional version of Thm. 1.5, we can conclude that there exists $T_m \in B(\Gamma_s(H_m^{\perp}))$ such that $U_m^* T U_m = 1 \otimes T_m$.

Now

$$x = Te(0) = U_m(1 \otimes T_m)U_m^* e(0) = U_m(1 \otimes T_m)(e(0) \otimes e(0)) = U_m(e(0) \otimes T_m e(0)).$$

Thanks to Lemma 2.7, x_n is orthogonal to $\xi^{\otimes n}$. This proves the claim.

Let $\lambda \in \mathbb{C}$ be such that $Te(0) = \lambda e(0)$. Calculate as follows to observe that for $\xi \in H$,

$$\begin{aligned}
Te(\xi) &= e^{\frac{||\xi||^2}{2}} TW(\xi)e(0) \\
&= e^{\frac{||\xi||^2}{2}} W(\xi)Te(0) \\
&= \lambda e^{\frac{||\xi||^2}{2}} W(\xi)e(0) \\
&= \lambda e(\xi).
\end{aligned}$$

Hence, $T = \lambda$. The proof of Thm. 1.5 is now complete. $\square$

References

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